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BENCHMARK SOLUTIONS FOR TSUNAMI WAVE FRONTS AND RAYS.
PART 2: PARABOLIC BOTTOM TOPOGRAPHY

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Abstract. In this paper, the kinematics of the tsunami wave ray and the wave front in the area, where the depth increases proportional to the squared distance to the straight shoreline, is studied. The exact analytical solution for the wave-ray trajectory above the parabolic bottom topography has been derived. This solution gives the possibility to determine in the ray approximation the tsunami wave heights in an area with the parabolic bottom topography. The distribution of the wave-height maxima in the area with such a bathymetry is compared to that obtained with a shallow-water model. All the exact solutions obtained can be used for testing numerical algorithms.

Key words: *tsunami propagation, shallow-water equations, wave ray, wave front kinematics*

1. SOME FEATURES OF THE LONG WAVE PROPAGATION

Tsunami waves, usually generated by vertical displacements of large ocean bottom areas, belong to a class of long waves whose length is at least ten times greater than the depth. The propagation of such waves in a deep ocean, where the wave height is usually two orders lower than the depth, is described by a linear system of differential shallow-water equations (Stocker, 1957). The validity of this description has many times been verified in practice. In the one-dimensional case without external forces (except for the gravity) these equations can be written down in the following form:

$$, \tag{1.1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Du)}{\partial x} = 0 . \tag{1.2}$$

Here u is the horizontal water flow velocity in the wave, η is the water surface height above an unperturbed level, g is the acceleration of gravity, and D is the depth. It follows from the shallow-water equations that the wave velocity does not depend on its length, and is determined by the Lagrange formula (Stocker, 1957):

$$c = \sqrt{gD} . \tag{1.3}$$

This formula is of fundamental importance for the kinematics of long waves (in particular, tsunamis). For the description the tsunami wave dynamics in the coastal zone, where the tsunami amplitude increases and the depth decreases, the nonlinear shallow water model is used (Marchuk et al., 1983). The wave propagation velocity for this model is expressed by the formula

$$c = \sqrt{g(D+\eta)} . \tag{1.4}$$

For the linear system of shallow water equations the horizontal flow velocity in a moving wave is expressed by the formula (Marchuk, 2016)

$$u = \eta \sqrt{\frac{g}{D}} , \tag{1.5}$$

and for the nonlinear tsunami wave formula (1.5) will be changed to

$$u = \eta \sqrt{\frac{g}{D+\eta}} . \tag{1.6}$$

Independence of the tsunami front propagation velocity on the wave parameters (height and length) gives the possibility for a priori discovery of peculiarities of the wave behavior in areas with uneven bottom. In the paper (Marchuk, 2016) the formula for the 1-D wave height varying in the moving wave has been derived.

In the linear case it can be written as

$$\eta_2(x) \approx \eta_1(x)\sqrt[4]{\frac{D_1}{D_2}}\,, \tag{1.7}$$

where η_2 and D_2 are the current wave elevation and depth and η_1 and D_1 are the initial values. It is the well-known Green formula describing a height variation of a long wave over an uneven bottom in the one-dimensional case. If a wave front is not straight, the wave amplitude varies not only due to the non-constant depth, but also as a result of wave refraction, that is, the wave-front line transformation. In the same paper (Marchuk, 2016) the relation between the wave segment length and the amplitude of this wave segment was also obtained

$$\eta_2 = \eta_1\sqrt{\frac{L_1}{L_2}}\,. \tag{1.8}$$

Here L_2 is the current length of the wave segment and L_1 is its initial length. Thus, due to the cylindrical propagation the tsunami wave height decreases inversely proportional to the square root of the circular front radius or the wave front length. In general, the kinematics of propagation of perturbations in various media is described by the eikonal equation. The governing formulas for the wave-front kinematics are presented in (Romanov, 1984), where the wave ray concept one of whose properties is the orthogonality to the wave-front line at any time is introduced. Along wave rays, a perturbation propagates from a source to other points of the medium in the least time. This means that wave rays are the fastest routes. Between the two closely spaced wave rays (in a ray tube), the wave energy remains constant (Romanov, 1984). Therefore, for a wave segment in a ray tube, formulas (1.7) and (1.8) can be rewritten in the form

$$\eta_2 = \eta_1\sqrt{\frac{L_1}{L_2}}\sqrt[4]{\frac{D_1}{D_2}}\,, \tag{1.9}$$

where L_1 and L_2 are the widths of the ray tube (the length of the wave-front line segment inside the ray tube) at the initial and current time moments of wave propagation.

2. EXACT ANALYTICAL FORMULAS FOR WAVE-RAY TRACES ABOVE THE PARABOLIC BOTTOM TOPOGRAPHY

An exact mathematical formula for a wave ray trajectory over a parabolic bottom can be found from the laws of geometrical optics. Let us consider a two-dimensional water area where the depth and the wave propagation velocity vary only in one direction. In this case we can use the Snell law for the wave ray refraction angle in a medium with varying optical conductivity (Sabra, 1981). According to this law, if in a two-dimensional conducting medium a ray comes at the angle of incidence α_1 to the horizontal line (Fig. 2.1), and the conductivity (the propagation velocity of a signal) changes from b_1

to b_2, then after passing the interface its direction α_2 changes according to the formula

$$\frac{\sin(\alpha_1)}{b_1} = \frac{\sin(\alpha_2)}{b_2}. \tag{2.1}$$

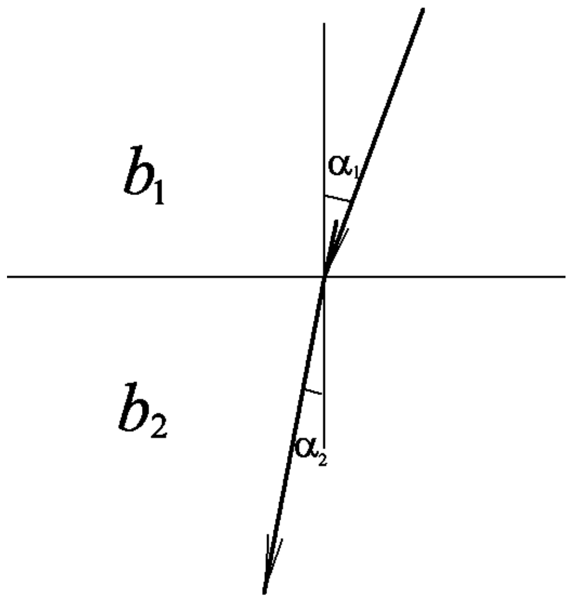


Figure 2.1. Refraction of a wave ray at the interface between two media

Thus, in a medium where the conductivity b (the wave propagation velocity) varies only along one spatial variable (for instance, $b(y)$), the wave ray inclination from the direction of the conductivity gradient α changes according to the formula

$$\frac{\sin(\alpha(y))}{b(y)} = const1 = \frac{\sin(\alpha_0)}{b(y_0)}. \tag{2.2}$$

Here α_0 is the initial incidence angle of the wave ray with respect to the vertical at the point $y=y_0$. In the case of a parabolic bottom, where the depth is proportional to the squared medium conductivity $a(y)$ (the tsunami wave propagation velocity) can be determined by Lagrange's formula (1.3), which for a parabolic bottom topography has the following form:

$$a(y) = \sqrt{g \cdot b_1 \cdot y^2} = b_2 \cdot y. \tag{2.3}$$

Here g is the gravity acceleration, y is the distance to the straight coastline where y=0, b_1 and b_2 are constant in the whole area. In this case the Snell law (2.2) gives the following wave-ray direction α according to the coastline normal

$$y(\alpha) = b_3 \cdot \sin(\alpha), \tag{2.4}$$

where b_3 is also a constant.

In order to determine the wave ray trajectory above such a bottom topography let us consider the following problem. Let the point (0, y_0) be a starting point for the wave ray that exits this point in parallel to the shoreline direction (Fig. 2.2). At the starting point, the angle $\alpha = \pi/2$. The inclination angle of a wave ray $y(x)$ to the X-axis will be expressed as $\beta = \pi/2 - \alpha$ (Fig. 2.2). Hence, we have

$$\frac{dy}{dx} = tg(\beta) = \frac{\sin(\beta)}{\cos(\beta)}, \qquad 0 < \beta < \pi/2$$

or

$$dx = dy\frac{\cos(\beta)}{\sin(\beta)}. \tag{2.5}$$

From (2.4) and (2.5) follows

$$dx = b_3 \cdot \sin(\beta) \cdot d\beta \frac{\cos(\beta)}{\sin(\beta)} = b_3 \cdot \cos(\beta) \cdot d\beta .$$

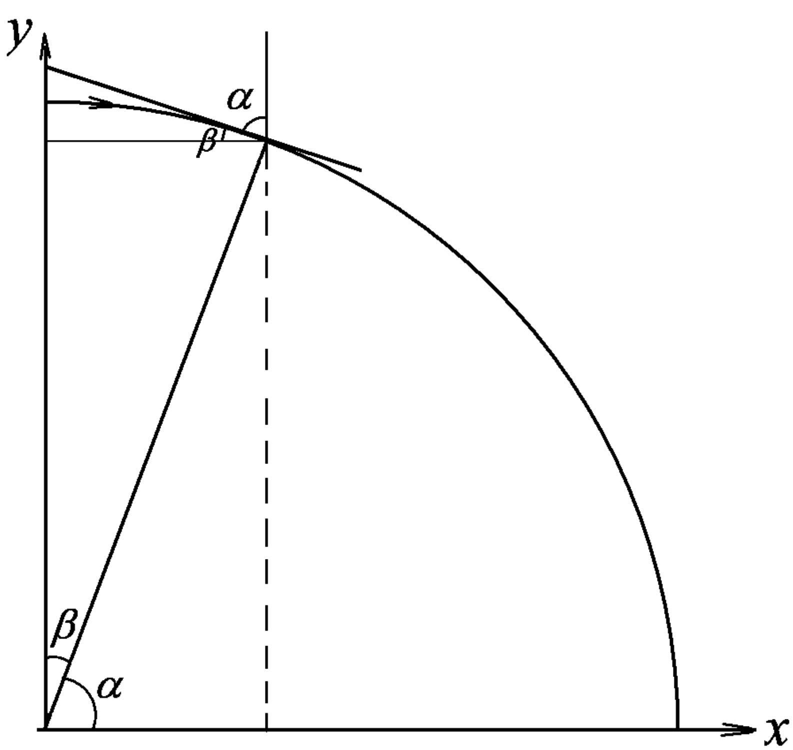


Figure 2.2. The wave ray refraction over the parabolic bottom topography

Assuming that along the wave ray the angle β varies from zero to the positive value β_1, then after conducting the integration we obtain

$$x = b_3 \cdot \sin(\beta)\Big|_0^{\beta_1} = b_3 \cdot \sin(\beta_1) = b_3 \cdot \cos(\alpha_1), \tag{2.6}$$

where $\alpha_1=\pi/2-\beta_1$. In addition to this, as follows from (2.4) for any value of α_1 from the interval $(\pi/2, 0)$

$$y = b_3 \cdot \sin(\alpha_1). \tag{2.7}$$

Formulas (2.6), (2.7) present the parametric equation of the circle of radius b_3 with a center located at the coordinate origin. This radius can be easily determined from (2.4). From this formula it also follows that the circle center is always situated on the X-axis. In the case when the initial wave-ray outgoing direction was in parallel to the shoreline, the circle radius is equal to the offshore distance at this moment. If at some time instance the angle between the ray and the Y-axis is equal to α_0 and the offshore distance at this moment is equal to y_0, then the radius of the circle which presents the wave ray can be found from (2.4)

$$r = b_3 = \frac{y_0}{\sin(\alpha_0)}. \tag{2.8}$$

If $\alpha_0=0$, but at the same time $y_0>0$, then the radius will be infinitely big and the ray trajectory will be presented by the straight line which is orthogonal to the coastline.

Unlike the case with a sloping bottom (Marchuk, 2016) here the boundary value problem for a wave ray can be solved without difficulty. Let us have two points (a source and a receiver) (2.3) in the area with the parabolic bottom topography. Let the receiver be situated at the coastline in the coordinate origin (0, 0), and the source coordinates be the following: $x=x_0$, $y=y_0$. Let, for definiteness, $0<x_0<y_0$, which means that a wave ray monotonically approaches the shore (Fig. 2.3).

Taking into account the fact that the wave ray is presented by the circle arc having the center at the shoreline (y=0), the unknown radius r can easily be found from the equation of the circle passing the point (x_0, y_0) and the coordinate origin

$$(r - x_0)^2 + y_0^2 = r^2. \tag{2.9}$$

Resolving this equation for the parameter r, the following expression will be written

$$r = \frac{x_0^2 + y_0^2}{2x_0}. \tag{2.10}$$

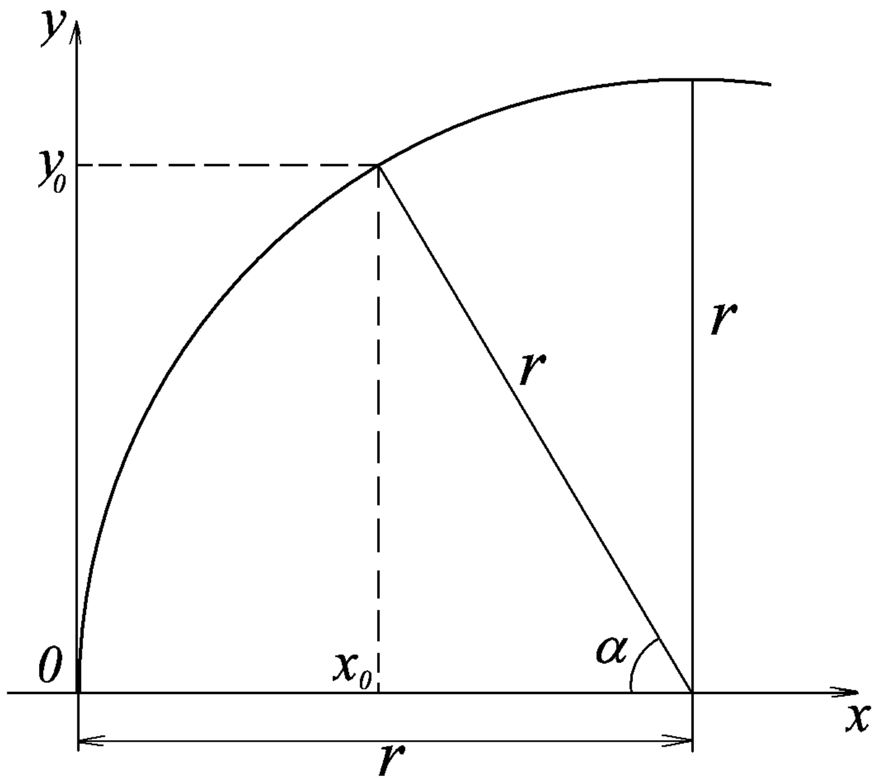


Figure 2.3. The scheme of solving the boundary-value problem for a wave ray above the parabolic bottom topography

Then from the equation of a circle in parametric form

$$(r-x)^2+y^2=r^2,$$

the equation of the wave ray passing the point (x_0,y_0) and the coordinate origin can be written down as

$$y=\sqrt{x(2r-x)}\ , \qquad (2.11)$$

where the circle radius r is determined by formula (2.10).

Now with the help of the previously derived formulas let us define the tsunami travel time along the wave ray. Let us consider the wave ray presenting an arc of the circle of radius r with the center located at the point (x_0,0) (Fig. 2.4). Let initially (when t=0) the wave front cross this wave ray at a point **A** with the coordinates (x_1, y_1). In addition, the segment which connects the point **A** with the center of the circle is inclined at the angle α_1 to the X-axis (Fig. 2.4).

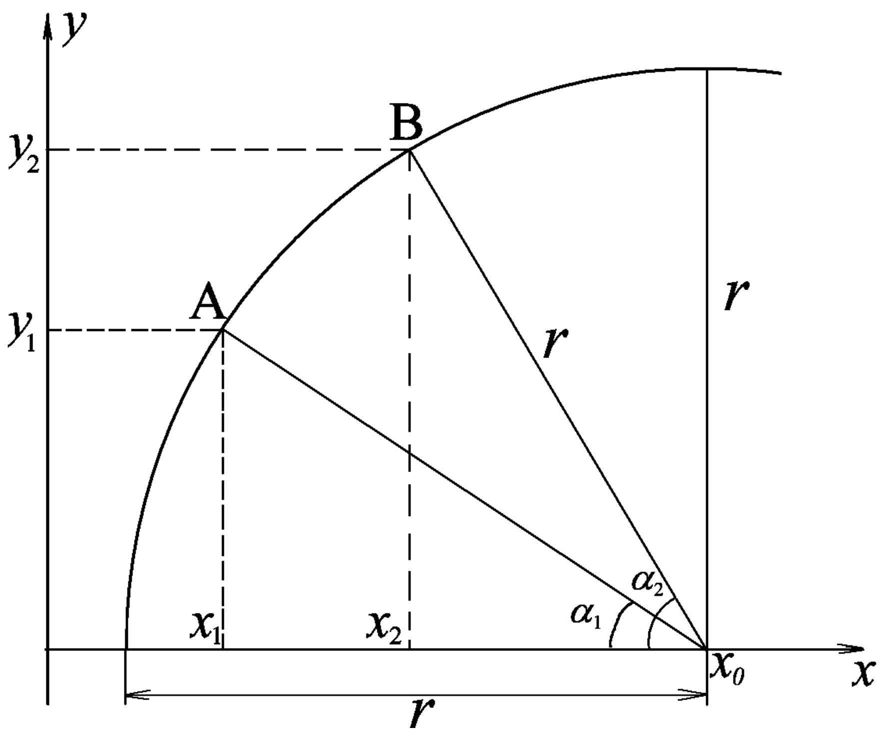


Figure 2.4. The scheme of the travel-time calculation along the wave ray between the points **A** and **B**

Let us find a travel time T which is required for the tsunami wave to arrive at the point **B** with the coordinates (x_2, y_2) and where the radius-vector inclination is equal to α_2 radians. If $0 < \alpha_1 < \alpha_2 \leq \pi$, then with allowance for (1.3) we can express the travel time as the Fermat integral

$$T = \int_{\alpha_1}^{\alpha_2} \frac{d\alpha \cdot r}{\sqrt{gD(x,y)}} \tag{2.12}$$

The depth D all around the area varies according to the formula

$$D(x,y) = k^2 y^2 . \tag{2.13}$$

When we come to the variable α, then (2.12) can be rewritten as

$$T = \frac{1}{\sqrt{g}} \int_{\alpha_1}^{\alpha_2} \frac{d\alpha \cdot r}{k \cdot r \cdot \sin(\alpha)} = \frac{1}{k_1} \ln \left| tg\left(\frac{\alpha}{2}\right) \right| \Bigg|_{\alpha_1}^{\alpha_2} = \frac{1}{k_1}\left(\ln\left| tg\left(\frac{\alpha_2}{2}\right)\right| - \ln\left| tg\left(\frac{\alpha_1}{2}\right)\right| \right), \tag{2.14}$$

where $k_1 = k \cdot \sqrt{g}$. The angle α is counted clockwise from the X-axis. From (2.14) it is possible to express the angle α_2 through the angle α_1 and the time T

$$\alpha_2 = 2arctg\left(\exp(k_1 T) \cdot tg\left(\frac{\alpha_1}{2} \right) \right). \tag{2.15}$$

Finally, the coordinates of the destination point which the wave front will reach at the time T going along the wave ray, can be presented as

$$x_2(T, \alpha_1) = x_0 - r \cdot \cos(\alpha_2),$$
$$y_2(T, \alpha_1) = r \cdot \sin(\alpha_2). \qquad (2.16))$$

Here α_2 is associated with T and α_1 according to (2.15). If $0 \le \alpha_2 < \alpha_1 < \pi/2$, then the expression for α_2 (2.15) will be as follows

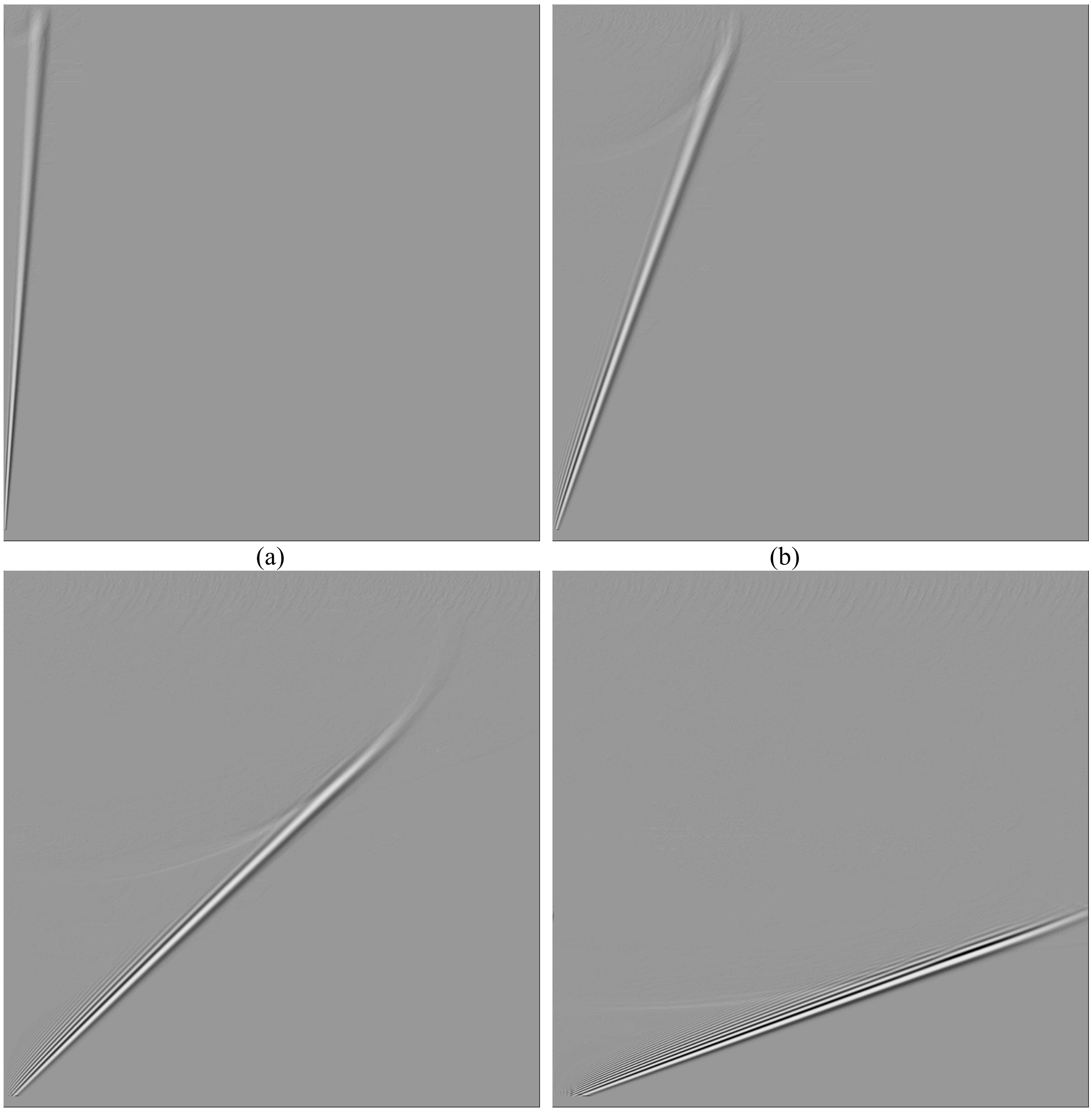
$$\alpha_2 = 2arctg\left(\frac{tg\left(\frac{\alpha_1}{2} \right)}{\exp(k_1 T)} \right). \tag{2.17}$$

The radius of this circle and its center position are uniquely determined from the source coordinates (x_1,y_1) and the exit angle α_1 of the wave ray (Fig. 2.4)

$$r = y_1 / \sin(\alpha_1),$$
$$x_0 = x_1 + r \cdot \cos(\alpha_1) = x_1 + y_1 / tg(\alpha_1). \qquad (2.18)$$

If $\alpha_1 = \pi/2$, then the radius r is certainly equal to y_1, and the abscissa of the circle center is the same as the one for the ray exit point (i.e. x_1). Now, using formulas (2.15) and (2.16) it is easy to determine coordinates of the destination point (x_2, y_2) located on the ray, where the wave front arrives at the time instance T.

It follows from formula (2.14) that the travel time is not dependent on the circle radius but only on the sector limits. This means that the travel time along the circle arc between the radius inclination angles α_1 and α_2 (Fig. 2.4) is the same for any concentric circle and the initially straight segment of a wave front will conserve its straight shape in the course of propagation over the parabolic bottom relief. An example of such a behavior of the straight wave front is shown in Figures 2.5 (a)-(d). Here the snapshots of the simulated tsunami wave (the water surface elevation), which propagates off the left boundary of the computational domain, are presented. It is easy to see that the wave front has the straight shape during whole propagation process. Such a property of the initially straight wave front segment can be used for testing numerical methods for computing the tsunami kinematics and dynamics.



(a) (b) (c) (d)

Figure 2.5. The snapshots of the calculated water surface as a result of tsunami propagation off the left boundary.

3. DETERMINATION OF A WAVE-FRONT LINE AND ESTIMATION OF THE WAVE HEIGHT ABOVE THE PARABOLIC BOTTOM

For some model shapes of the bottom, distributions of wave amplitudes (heights) can be found analytically. Consider, for example, the coastal area where the depth increases with a squared distance to the coast with a model tsunami source in the form of a circle of radius R_0 with the center at a distance of y_{00} from a straight coastline. In Figure 3.1, this line coincides with the axis OX ($y = 0$). In Section 2, the wave ray trajectory over a parabolic bottom (as in the case in question) has already been found. In order to determine a tsunami wave height all around the domain with the parabolic bottom topography, let us split it to many ray tubes varying the ray starting points and determining exiting angles there. Then using formulas (2.15) and (2.16) we can find coordinates of the points along each wave ray varying the time parameter T. The ray radius and the abscissa x_0 of the circle center are given by (2.18).

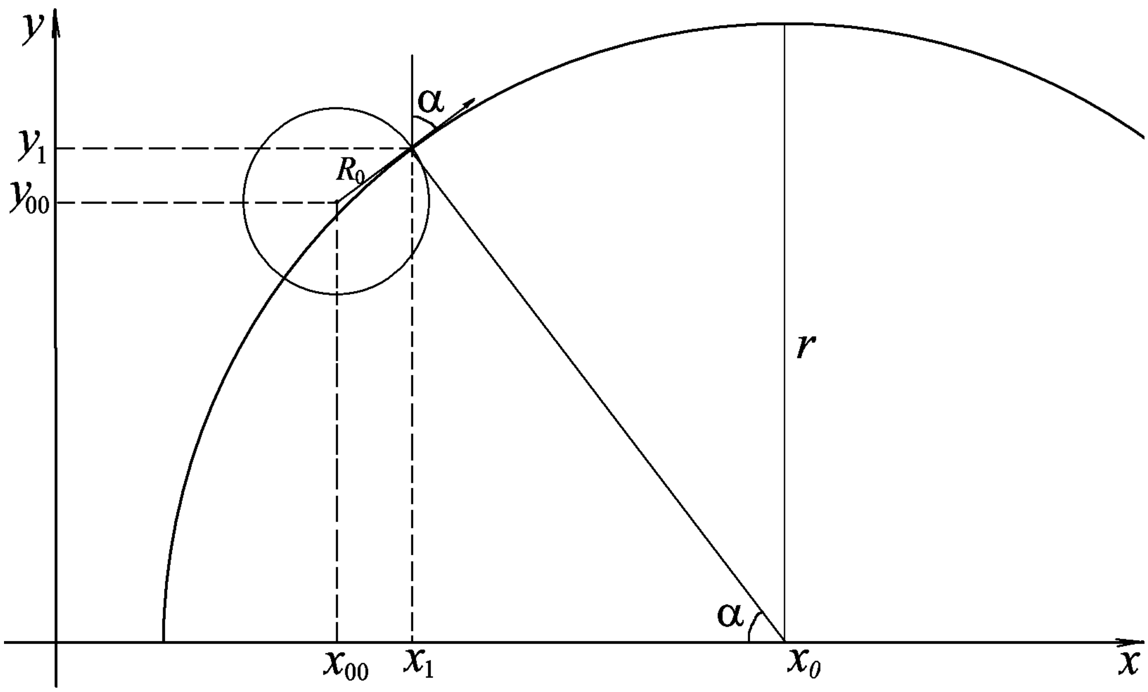


Figure 3.1. The wave-ray trace above the parabolic bottom which exits the circled source boundary point within the angle α relating to the ordinate axis

If, for example, a tsunami source is presented by a circle of radius R_0 with its center located at the point (x_{00},y_{00}), then N wave rays start from the source boundary points in the radius-vector directions. Then we will obtain a set of N wave rays coming from the source boundary up to the edges of the computational domain. The scheme of constructing such a ray is shown in Figure 3.1. We will split the whole domain to N-1 ray tubes by constructing wave rays exiting N different points, which are equidistantly located around the source. Here it is necessary to take into account the fact that coordinates of the ray starting points (x_i,y_i) vary according to the formulas

$$x_i = x_{00} + R_0 \cdot \sin(\alpha_i)\,,$$
$$y_i = y_{00} + R_0 \cdot \cos(\alpha_i)\,. \qquad (3.1)$$

Here α_i is the ray exiting angle according to the ordinate axis. It is not allowed to say that rays reach the shoreline, because from (2.14) it follows that the required travel time for this is infinitely long. Figure 3.2 shows the wave-ray set, which was built using 200 ray exiting points along the circled source boundary. Their exiting angles α_i were equal to $i \cdot \pi/200$ (i=1,…200). Thus, we have obtained the coordinates of the destination point versus the time *T* and the angle α. Now, with formulas (2.15) - (2.17) we can find the wave front location by fixing the travel time *T*.

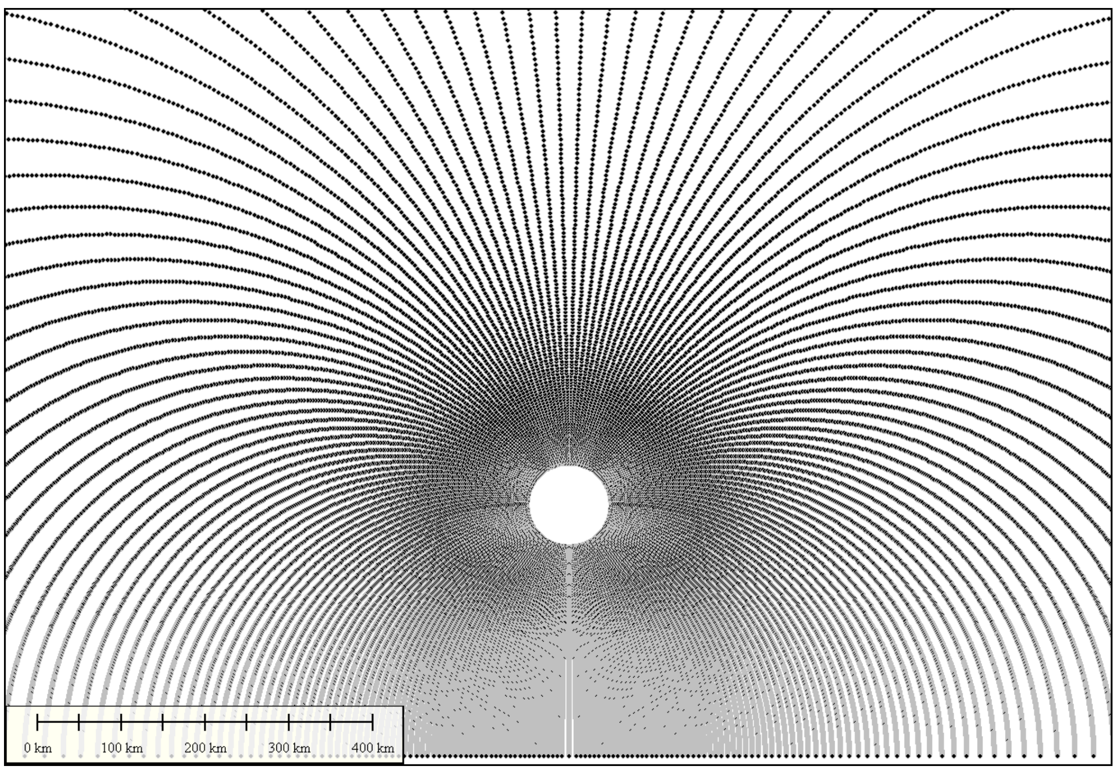


Figure 3.2. Wave-ray traces above the parabolic bottom topography coming from 200 points locating at the circled source boundary

If we draw the lines connecting the points along wave rays corresponding to the same time instance, we will obtain tsunami isochrones. As an example, Figure 3.3 shows positions of the wave front within 5-minutes interval. In this case, the center of the circled source of radius R_0=50 km was situated 300 km off the straight shoreline. Here the coefficient *k* of the parabolic depth growth (2.13) is equal to 10^{-4}. This means that at a distance of 1,000 km off the shore the depth is equal to 10,000 m.

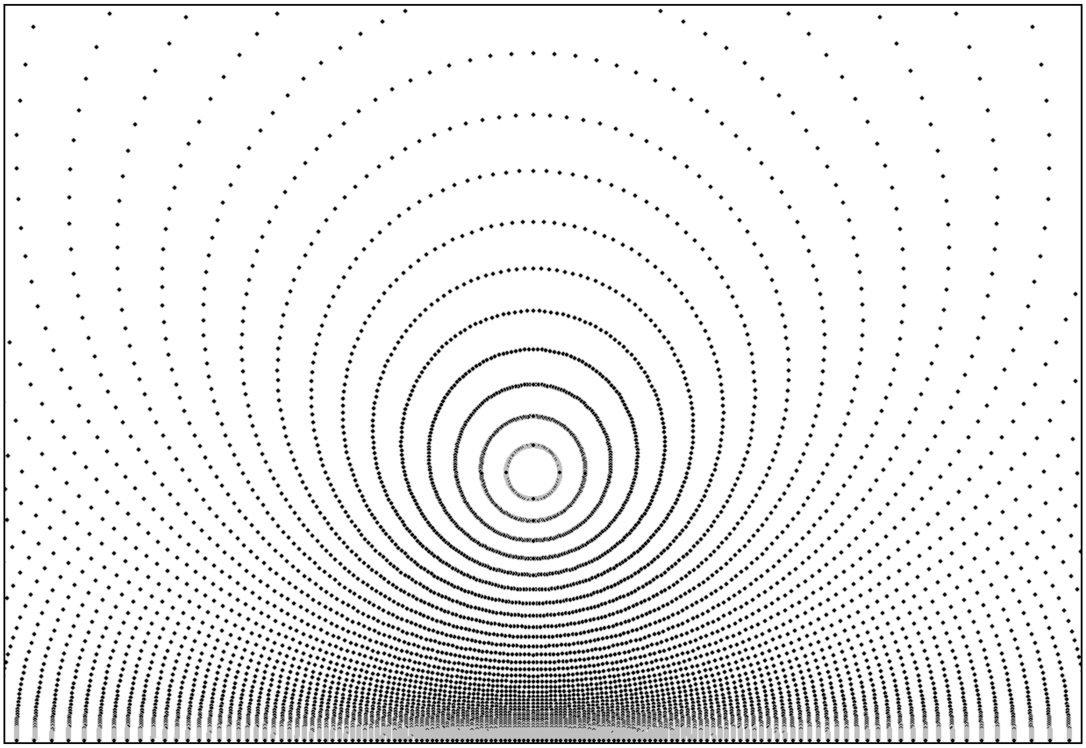


Figure 3.3. Positions of the tsunami wave front (isochrones) within the 5-minutes interval from a circled source of radius 50 km above the parabolic bottom topography

It is possible to prove that over the parabolic bottom topography, the initially rounded wave front will conserve its circle shape at any time instance after the beginning of propagation (Borovskikh, 2006). However, the circle center position will migrate off the shoreline (the lower boundary in Fig. 3.3). If the initial wave front is presented by the circle of radius R_0 with the center located at the distance y_{00} offshore then at a time instance T the circle-shaped wave front radius R_2 and position of its center y_2 can be expressed by formulas

$$R_2 = \left((y_{00} + R_0) \cdot e^{kT\sqrt{g}} - \frac{(y_{00} - R_0)}{e^{kT\sqrt{g}}} \right) / 2 \, , \qquad (3.2)$$

$$y_2 = \left((y_{00} + R_0) \cdot e^{kT\sqrt{g}} + \frac{(y_{00} - R_0)}{e^{kT\sqrt{g}}} \right) / 2 \, . \qquad (3.3)$$

Here the bottom inclination coefficient k is defined by (2.13). This is one more benchmark solution which can be used for testing numerical methods.

If we want to estimate the wave height at the point (x_1, y_1) it is necessary to determine the distance between the two following points: the first one is the point (x_1, y_1), where the wave going along the ray exiting the point (x_{00}+R_0 sin α, y_{00}+R_0 cos α) at the angle α (fig. 3.1) arrives at the time T. The second one is the point (x_2, y_2), where a tsunami wave arrives at the same time moment going along the wave ray exiting the point (x_{00}+R_0 sin(α+Δα), y_{00}+R_0 cos(α+Δα)) at the angle α + Δα. With formula (1.9), the coefficient of wave attenuation due to changing the ray tube width and depth is calculated. Doing this for various values of the time T and the directions of wave rays, we obtain the wave attenuation distribution over the entire area of points which can be reached by the wave rays from the initial wave front points.

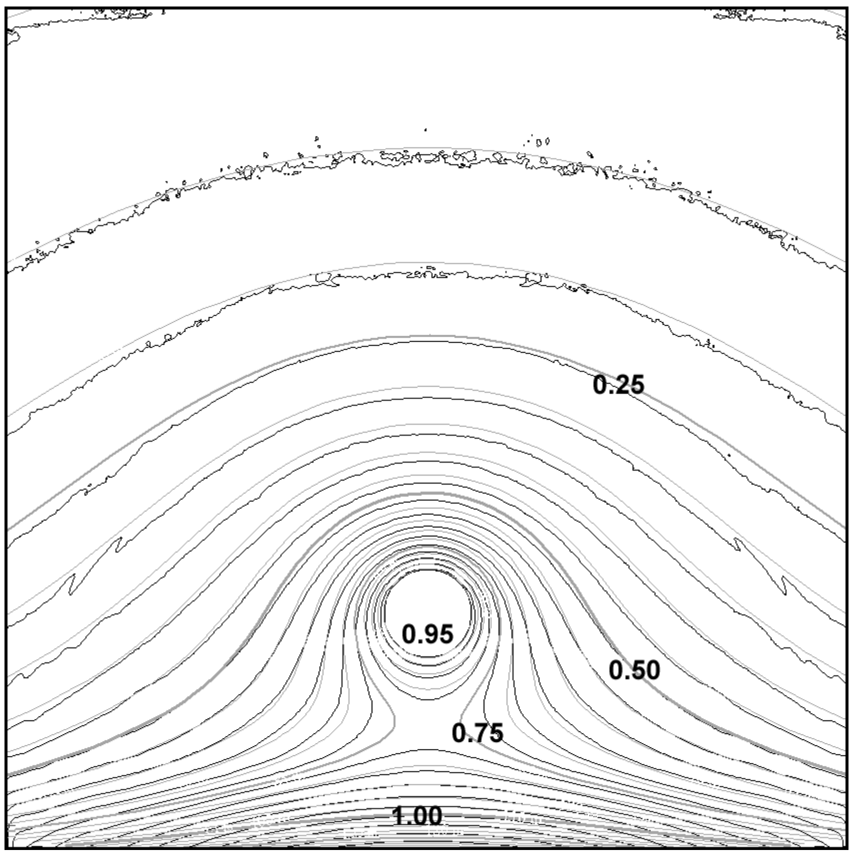


Figure 3.4. The comparative location of the wave-height maxima isolines obtained by numerical calculation of the shallow-water equations (black color) and within the wave-ray approximation (grey color)

To verify the solution obtained, the numerical simulation of the tsunami wave propagation was carried out using the differential shallow-water model with a package called MOST (Titov, 1997). In this numerical experiment the center of a circular source, 40 km in radius, was located at a distance of 300 km from the coast. This source formed a 100-cm high circular wave at a distance of 50 km from the center. This initial front was taken as initial conditions to calculate the amplitudes with the ray model. In Figure 3.4, isolines of the tsunami wave height maxima in the 1000 × 1000 km coastal area with a parabolic bottom obtained from formulas (2.15)–(2.16) and (1.9) are shown by grey color. For comparison, isolines of the wave height maxima obtained by numerical solution of the same problem with the nonlinear shallow water model (Titov, 1997) having the same initial values are shown by black color. In both cases, the levels of isolines (whose height is shown in meters), were taken with a spacing of 5 cm. Figure 3.4 shows that the distributions of amplitudes obtained by the two different methods mostly coincide except the shelf area, where a depth is less than 150-200 m where, in contrast to the ray approximation, in the numerical implementation of the differential shallow water model the influence of the nonlinearity is much stronger than in a deep sea.

The parabolic bottom topography (or similar) can be found in several tsunamigenic regions. As an example, the cross-section of the land-bottom relief (Mikheeva et al., 2014) of the Fukushima-Tohoku area (Japan) is presented in Figure 3.5. The white line in this figure shows the cross-section direction.

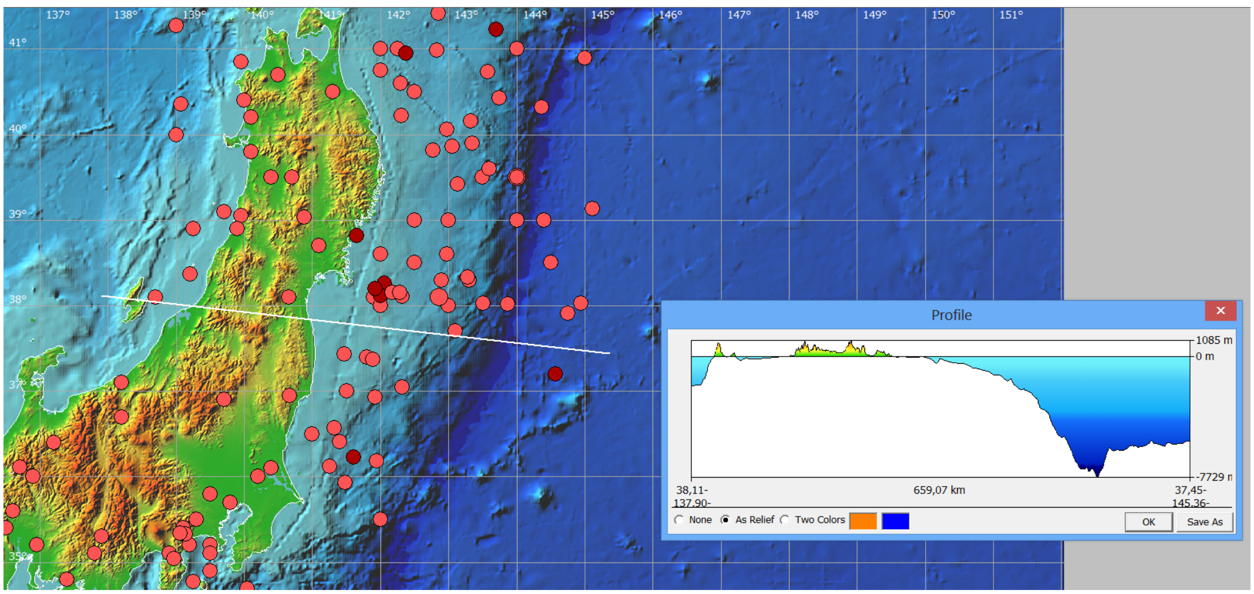


Figure. 3.5. The cross-section of the land-bottom relief through the north of Honshu Island (Japan).

Moreover, one can see that northwards of this cross-section such a kind of bottom topography (parabolic-type) takes place up to the top of Honshu Island.

CONCLUSION

The height of a propagating tsunami wave versus depth and refraction above an uneven bottom has been estimated using the differential shallow-water equations. The exact wave ray trajectory and the tsunami isochrones above the parabolic bottom have been found. A comparison of the results obtained by the ray method and with the shallow water model has been made. This comparison shows that with a numerical method based on the ray approximation not only the arrival times of tsunami waves at different points, but also the wave heights in a deep water can be estimated. These solutions for travel times and wave-ray traces can be used for testing the numerical methods carrying out the tsunami simulation.

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