THE ALL-SOURCE GREEN'S FUNCTION AND ITS APPLICATIONS TO TSUNAMI PROBLEMS

ZHIGANG XU

CANADIAN HYDROGRAPHIC SERVICE, MAURICE LAMONTAGNE INSTITUTE
FISHERIES AND OCEANS CANADA

P.O. Box 1000, 850 route de la Mer Mont-Joli, Québec Canada G5H 3Z4

E-mail: xuz@dfo-mpo.gc.ca

ABSTRACT

The classical Green's function provides the global linear response to impulse forcing at a particular source location. It is a type of one-source-all-receiver Green's function. This paper presents a new type of Green's function, referred to as the all-source-one-receiver, or for short the all-source Green's function (ASGF), in which the solution at a point of interest (POI) can be written in terms of global forcing without requiring the solution at other locations. The ASGF is particularly applicable to tsunami problems. The response to forcing anywhere in the global ocean can be determined within a few seconds on an ordinary personal computer or on a web server. The ASGF also brings in two new types of tsunami charts, one for the arrival time and the second for the gain, without assuming the location of the epicenter or reversibility of the tsunami travel path. Thus it provides a useful tool for tsunami hazard preparedness and to rapidly calculate the real-time responses at selected POIs for a tsunami generated anywhere in the world's oceans.

Keywords: all-source Green's functions, real-time tsunami arrival simulations, tsunami arrival time charts, tsunami gain charts.

1. INTRODUCTION

Linear long wave dynamics can adequately describe tsunami propagation in the deep ocean (> 50m, Shuto 1991). For a linear dynamic system, the Green's function, defined as the system response to isolated and impulse external forcing, is fundamental. This is because any other type of external forcing can be represented as a sum of isolated impulses and, because of the linear superposition principle, the system response to arbitrary forcing is then merely a linear combination of the Green's functions. The Green's function can be pre-calculated for fixed physical parameters and geometry of the system. The pre-calculation implies a great time saving, which is essential when a catastrophe occurs and a rapid solution is required.

Due to the complexity of realistic systems, the Green's function for most problems of interest requires numerical calculation. Since impulse forcing at one time step is equivalent to an initial condition at the next time step, a Green's function is traditionally obtained by setting up a unitary initial condition at a model grid point (source point) and then running a linearized model to get a solution field defined everywhere. This is a type of one-source-all-receiver Green's function, or briefly, one-source Green's function.

Since external forcing may happen anywhere and with any spatial distribution, one would have to run the model repeatedly by taking every grid point successively as the source point to achieve thorough hazard preparedness. If there are N grid points in total, and if one defines a run for one-source-all-receiver as a $1 \times N$ problem, then the total of all runs will be an $N \times N$ problem, which is infeasible to tackle when N is large. Consequently, such repeated runs must be limited to pre-assumed source regions, e.g., a seismic active zone (a limited-source Green's function approach).

This paper presents a new type of Green's function, an all-source-one-receiver Green's function. For convenience, it may be alternatively referred as an all-source Green's function (ASGF). The ASGF focuses on a receiver point, regarding all the model grid points as the potential sources. One may freely choose a point of interest (POI), for socio-economic, academic, or monitorial reasons, as a receiver. A run for N-sources *vs.* 1-receiver is an $N \times 1$ problem and, as will be shown in the next section, its computational cost is the same as for the $1 \times N$ problem mentioned above.

The ASGF can be used as a tool to rapidly provide responses at one or several POIs for the full range of possible tsunami source locations; it allows almost instantaneous (within a few seconds) determination of the linear response at one or several POIs to a tsunami originating anywhere in an ocean basin or even in the global ocean. We shall see that it also invites the introduction of two new types of tsunami charts, one specifying the arrival time and the other specifying the gain, without assuming the location of an epicenter or the reversibility in the travel path of a tsunami. The next section will give the basic algorithm and the mathematical definition of the ASGF. Its potential application to the tsunami problem is illustrated in Section 3 and closing remarks are given in Section 4.

2. THE ALL-SOURCE GREEN'S FUNCTION (ASGF)

For a linear system, one can write the evolution equation for its state vector, X, in the general form

$$X^{(k)} = \mathbf{A}X^{(k-1)} + \mathbf{B}F^{(k-1)} \tag{1}$$

where $k = 1, 2, 3, \cdots$, indicate the time steps, **A** is the system matrix which encapsulates the model physics, **B** is the input matrix which projects the external forcing vector, F, onto the state vector. Once a tsunami is initiated by an earthquake or a submarine landslide, its evolution is determined by free wave propagation. This paper has two purposes, to introduce the ASGF and to demonstrate its usefulness to tsunami problems. For these purposes, the external forcing term will not be required; therefore, only the free motion problem is considered here. However, it is worth noting that the ASGF could also be used for a forced motion problem, e.g., a storm surge; in that case, continuous forcing must be considered and the second term would require additional consideration. The application of the ASGF to a storm surge problem will be considered in a future paper.

With the second term omitted, Eq. (1) can be re-expressed as

$$X^{(k)} = \mathbf{A}^k X^{(0)} \tag{2}$$

where $X^{(0)}$ is the initial state vector, $\mathbf{A}^k$ represents the *kth* power of the system matrix (in this paper, a superscript k refers to a time step if enclosed in parentheses and to a power otherwise). Eq. (1) (without the second term) and eq. (2) are mathematically equivalent, but can differ dramatically in their computational loads. If the solutions at all model grid points are required, eq. (1) would be the choice for computational efficiency, since each time step only involves a multiplication of a matrix with a vector. In this case, to choose eq. (2) would be unwise, since, when $k \geq 2$, it would involve calculating all the *kth* powers of the whole $N \times N$ matrix **A**, which would be very expensive or infeasible when N is large. On the other hand, if only a few elements of the state vector X need to be determined (i.e., only a few model grid points are interested for having solutions), eq. (2) provides a distinct advantage as shown below.

Assume that there is only one point, say the *ith* point, where the solution is required. From eq. (2), one can write the solution at the *ith* point as

$$X^{(k)}(i) = \mathbf{A}^k(i,:)X^{(0)} \tag{3}$$

where $\mathbf{A}^k(i,:)$ represents the *ith* row of the *kth* power of the system matrix, and the colon indicates all the columns of the matrix. The crux here is how to calculate $\mathbf{A}^k(i,:)$ economically. If one first calculates the whole matrix power, $\mathbf{A}^k$, and then extracts the *ith* row from it, the calculation is too expensive computationally. Can one obtain the value of just the *ith* row of the matrix power without calculating the other rows? The answer follows easily once the question is posed in this form. In fact,

$$G^{(1)} \equiv \mathbf{A}(i,:) \tag{4}$$

$$G^{(2)} = G^{(1)}\mathbf{A} \tag{5}$$

$$G^{(3)} = G^{(2)}\mathbf{A} \tag{6}$$

$$\vdots$$

$$G^{(k)} = G^{(k-1)}\mathbf{A} \tag{7}$$

$$\vdots$$

$$G^{(K)} = G^{(K-1)}\mathbf{A} \tag{8}$$

where the capital K represents the total number of time steps, $G^{(1)}$ is simply the *ith* row of matrix $\mathbf{A}$, and $G^{(k)}$ ($k=2,3, \cdots K$) are obtained by a series of multiplications of a row vector with the same matrix. As one can verify, $G^{(k)}$ is the *ith* row of $\mathbf{A}^k$ ($k=2,3, \cdots K$) but it is not extracted from the whole matrix to the *kth* power; it is calculated economically. The effects on the *ith* point of the coupling by all the other model grid points, as well as of the global topography and lateral boundaries, have been actually realized in the above vector-matrix multiplication sequence.

The series of $G^{(k)}$ define the ASGF, evaluated at $t = k\Delta t$ ($k=1,2,3, \cdots K$). Each $G^{(k)}$ is a row vector with N columns, each of which can be viewed as an information channel. Having N channels means that a receiver can simultaneously receive signals from N different sources without interference (think of $G^{(k)} = G^{(k)}\mathbf{I}$ where the identical matrix $\mathbf{I}$ contains N unitary sources); linearity will allow these individual contributions to be summed to obtain the full solution when required.

If we now form a new matrix $\mathbf{G}$ from the row vectors $G^{(k)}$ ($k=1,2,3, \cdots K$) as

$$\mathbf{G} = \begin{bmatrix} G^{(1)} \\ G^{(2)} \\ \vdots \\ G^{(k)} \\ \vdots \\ G^{(K)} \end{bmatrix} \tag{9}$$

then $\mathbf{G} \times X^{(0)}$ provides the solution time series at grid point i in response to an initial setup anywhere in the model domain. For a different initial setup, say, corresponding to a new tsunami, one can simply substitute a new column vector for $X^{(0)}$, without modifying the matrix $\mathbf{G}$. That is, once $\mathbf{G}$ has been determined, it can be repeatedly used; the evolution from any specified initial state is easily calculated from a simple matrix-vector multiplication. The determination of the matrix $\mathbf{G}$ may require significant computer resources, but the matrix-vector multiplication can be quickly performed even on an ordinary personal computer or on a web server.

The calculation of the aforementioned one-source-all-receiver Green's function will consist of a sequence of multiplications of the same matrix $\mathbf{A}$ and a column vector, which costs the same computationally as to calculate an all-source-one-receiver Green's function.

The ASGF approach does not require use of a particular model. All required is that the model dynamics are linear. Thus, one can always choose a better model. The ASGF will not alter the realism or quality of the particular model that it uses; it simply provides a very efficient means to quickly compute responses at particular POIs.

3. APPLICATION TO TSUNAMI PROBLEMS

This section will explore two aspects of the applications of the ASGF to tsunami problems, the real-time simulation for tsunami propagation, and two new types of tsunami prediction charts. For these applications, Heaps' two dimensional numerical sea model in spherical coordinates (Heaps 1969) is used, but rewritten in a matrix format so that the system matrix, $\mathbf{A}$, can be easily produced. The model spatial resolution is 5 min in both longitudinal and latitudinal, and the model topography is determined from a decimation of the Etopo 2 bathymetric data (http://www.ngdc.noaa.gov/mgg/fliers/01mgg04.html).

The examples presented here are influenced by artificial closed boundaries along 100° W, the prime meridian, the equator and 80° N. In addition, no tuning was done to improve the prediction skill of the model (e.g., by modifying friction parameters). Such improvements should clearly be implemented for practical applications, but for the demonstration purpose here, they are not essential.

3.1. Real-time tsunami propagation simulation

It is conceptually useful to think of the life of a tsunami as consisting of three phases: the genesis, the open-ocean propagation and the final onshore run-up. While the initial genesis and the final run-up are clearly nonlinear processes, the propagation through deep water fortunately obeys simple linear long wave dynamics. Nevertheless, real-time simulations for tsunami propagation are still lacking (Paul Whitmore, personal communication, 2005), principally because tsunamis move very quickly (~700 km/hour) across very large (transoceanic) domains. Even with a supercomputer and massively parallel computations, it is very difficult to win the race for time against a fast moving tsunami. The limited-source Green's function approach can provide results very quickly, but unfortunately only for cases in which a tsunami originates within source regions that are determined a priori.

The ASGF offers a new tool to address this challenge. Because it can be pre-calculated with the entire domain as potential source locations, when a tsunami-triggering event occurs anywhere within the domain, the part of the pre-calculated ASGF corresponding to the source region can be loaded into RAM, and its multiplication with the source function can yield almost instantaneously the tsunami arrival curves at one or several POIs.

Figure 1 and Figure 2 illustrate results from an R&D version of an internet-based real-time tsunami simulation system, accessible at http://odylab.qc.dfo-mpo.gc.ca . Figure 1 illustrates the graphical user interface provided in the form of a map on which one can create an arbitrary polygon defining the tsunami source region using a simple point and click procedure. Once this source region is specified, one has the option of checking one or several points of interest where a response time series is desired. Pressing the "Make plot" button then produces the required time series within a few seconds. For the example presented here, a 100 km × 100 km square was created at the position shown by the tiny white square near x=-4000 km and y=2000 km on Figure 1. Within 6 seconds of this source specification, a pop-up window showed the response curve at Halifax (Figure 2).

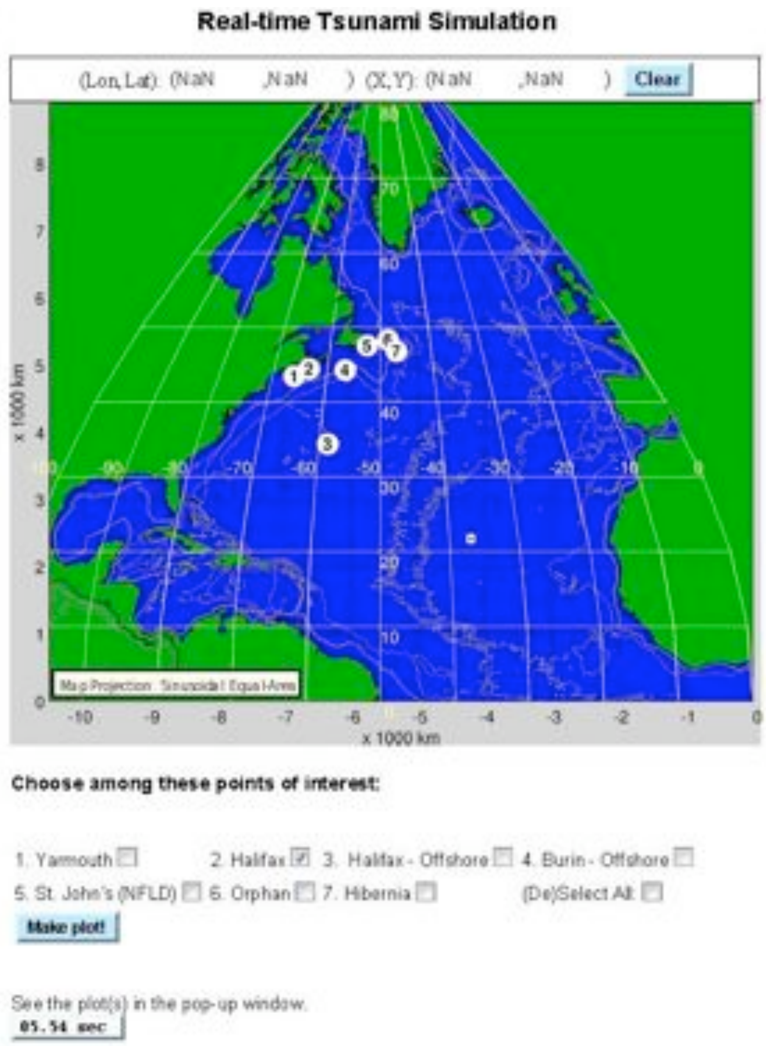

Figure 1. The graphical user interface for the tsunami simulation system. One may select points anywhere on the map to create a polygon defining the source region and then check one or more boxes as points of interest. Pressing the "Make plot" button then provides results such as those shown in Figure 2. A sinusoidal equal area map projection with a reference longitude of 50° W has been used.

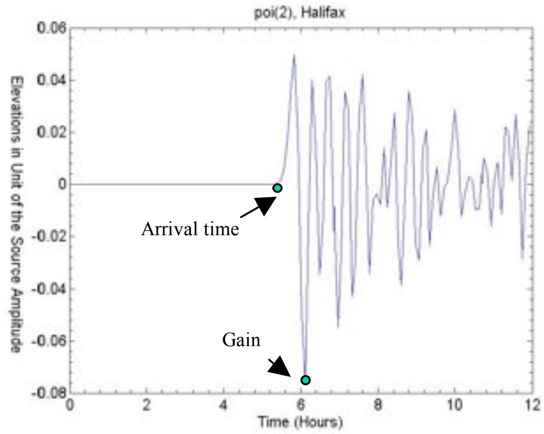

Figure 2. The tsunami arrival curve for the point of interest and the forcing region specified by the procedure described in the caption of Figure 1. Two pieces of information are particularly important, the first arrival time and the relative maximum amplitude.

3.2. New types of tsunami charts, the arrival time and the gain charts

As noted in Figure 2, two particularly useful pieces of information can be extracted from the response time series, the first arrival time of the signal and the largest relative amplitude during a certain period. The largest relative amplitude is defined as a gain since it is relative to the source unit. Proper definitions of the arrival time and gain can be discussed later. For demonstration purposes here, the arrival time is defined as the first time when a signal with 1/1000 relative amplitude (positive or negative) arrives, and a period of 12 hours is taken within which the maximum absolute relative amplitude is searched to determine the gain.

The 100 km × 100 km square in Figure 1 was purposefully chosen so that one can use it to tile the model domain. Each of the tiles will then give a pair of numbers for the arrival time and the gain. Contouring them gives an arrival time chart and a gain chart. Figure 3 and Figure 4 illustrate these charts for Halifax as the POI and the North Atlantic Ocean as the source field. In Figure 3, different color bands indicate arrival times at Halifax for disturbances generated anywhere over the model domain. Any tsunami originating in the same color band would arrive at Halifax within the same hour. The gain chart, Figure 4, gives the magnitude at any source point in the domain of the largest amplitude at the POI if a unit tsunami originates from that source point. It reveals a spatial structure indicating which potential generation locations would have maximal impact on a particular POI. Both charts are equally important. A combination of the gain map with a hazard map for earthquakes and submarine landslides should provide a useful tool for tsunami risk analysis.

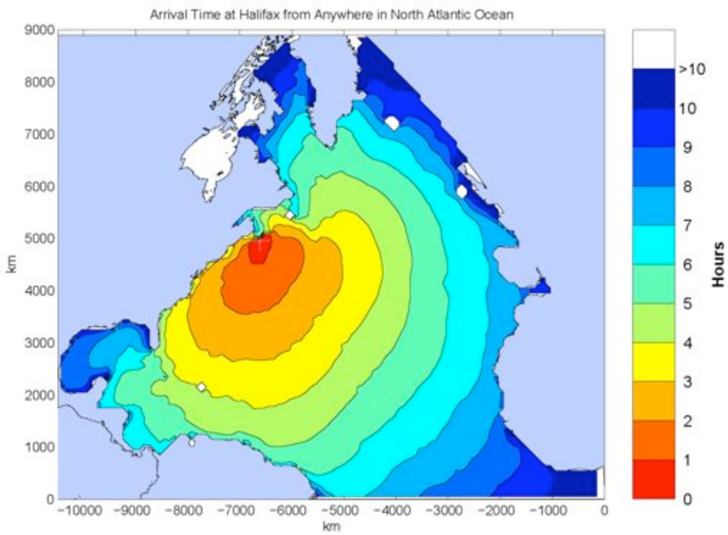

Figure 3. Arrival time map for Halifax as a POI. Any tsunami originating in the same color band would arrive at Halifax within the same hour.

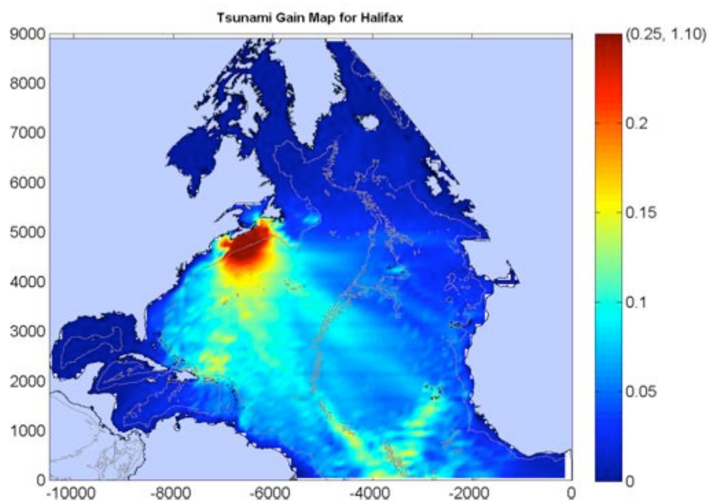

Figure 4. Gain map for Halifax. It gives the magnitude at any source point in the domain of the largest amplitude at the POI if a unit tsunami originates from that source point over a 100 km x 100 km region. It provides a spatial structure indicating which source regions would generate the largest responses at Halifax.

The tile size may, of course, be made as fine as the model spatial resolution, which is 5 min in both longitude and latitude for the model used to generate the results illustrated in this section ($\approx$10 km x 10 km cells in the equatorial region). However, since the gain is reduced for smaller tile size, the details of the map depend on this choice. A 100 km tile, with about 144 model grid points, was chosen here so that the gain map has both good spatial resolution and is still readable. The curve seen in Figure 2 is actually a summation of the contributions of these individual source points.

A traditional tsunami chart only contains information on the tsunami travel time, not on its amplitude. This is because it is made with an approach where a tsunami is treated as an optical ray from an assumed source. In this approach, one can deal with the time easily, but not so with the amplitude. Moreover, the usefulness of such a chart depends on whether a future tsunami will indeed originate from the assumed epicentre. To overcome this source dependence, one must also assume that a tsunami travel path is reversible like a light ray path and hence if one switches the source and receiver positions, the travel time is still the same. This reversibility assumption allows one to interpret the travel time from a POI to anywhere in the domain as the arrival time from anywhere to the POI, whereas the former can be obtained by placing a source simply at the POI and using the ray tracing technique. It has been the practice to use the ray tracing technique and the path reversibility assumption for making the tsunami charts (Paul Whitmore, 2006 and Tad S. Murty, 2006, personal communications; also see http://slowmo.sourceforge.net/).

It will be interesting to see how valid this path reversibility assumption is. On the one hand, we all know that the geostrophic component of long waves in the ocean prefer certain directions for their travel. For example, under the effect of the earth's rotation, they tend to travel in a direction with the shallow water to their right (left) hand side in the northern (southern) hemisphere (e.g., Pedlosky 1979). However, the purpose of this paper is not to conduct an expanded discussion on the path reversibility.

The arrival time and the gain charts made with the ASGF do not assume specific epicentres or the path reversibility. The Coriolis effect, the seabed frictional effects, and the wave refractions and reflections caused by arbitrary coastlines and topography are all accounted for.

4. CLOSING REMARKS

The all-source Green's function has been defined and an algorithm for its determination has been provided. Its usefulness in tsunami problems also has been demonstrated; it makes real-time simulation feasible and it introduces the arrival time and gain charts without assumptions about the epicentre location or the reversibility of tsunami ray paths.

In comparison with the traditional one-source-all-receiver Green's functions, the all-source-one-receiver Green's function has the following advantages: 1) The latter covers source points for the entire model domain whereas the former only covers one source point. 2) POIs are more easily identified than potential source regions; one may use a thousand POIs to cover comprehensively the important coastal cities worldwide, and perhaps a dozen POIs for an island country or region. In contrast, model grid points can easily number in the millions. To cover m POIs, the latter approach results in an $m \times N$ problem which is still linear in N (as long as $m \ll N$) and can be easily handled by modern computers.

The index i was taken to indicate a single POI in the discussion of the numerical algorithm, this was just for simplicity of presentation; nothing prevents use of the index i to represent several distinct POIs. One can thus use the same algorithm to calculate the ASGFs for several POIs simultaneously.

Also, the algorithm is completely ready for parallel computation; one can assign values of i to different processors or even different computational platforms and compute the individual solutions without any need for exchange of information between processors. It should be feasible to calculate and store the ASGFs for worldwide important cities as POIs and the entire global ocean as the potential sources. It would also be interesting to make a tsunami atlas consisting of the arrival time and gain charts covering the global domain.

The system matrix, A, should be very sparse, since it results from discretization of a set of partial differential equations. The Heaps's model adopted in this paper gives a sparsity of the order of 10^{-6} (the ratio of non-zero elements versus the total elements in the matrix). One should use a good sparse matrix technique, such as the nonzero-element-only technique implemented in Matlab (Gilbert *et al* 1992), for efficiency in the CPU time and in the disk storage.

The ASGF does not address the tsunami initial set-up or the final run-up issue but only addresses the tsunami propagation issue. However, it is the propagation issue that is the most computationally challenging. The ASGF provides an efficient intermediate connection between a genesis model and a run-up model at the two ends.

The initial state vector, $X^{(0)}$, contains both the elevations and the depth average velocities. This will allow for different types of initial conditions: the elevation only, or the currently only, or a mixture of them.

ACKNOWLEDGEMENTS

This work has benefited from personnel communications with Dr. Paul Whitmore and Tad S. Murty. Thorough internal reviews by Dr. Dan Wright, Brian Petrie, Denis Lefaivre and John Loder are greatly appreciated. The author also thanks Mr. Michel Beaulieu for his help in setting up the ODYLAB extranet site to demonstrate the real-time tsunami simulations. This work is partially supported by Fisheries and Oceans Canada's COMDA (Centre for Ocean Model Development and Application). It also benefits from another closely related project, Study of Water Level Variations and Modelling of Storm Surges in the Estuary and Gulf of St. Lawrence, co-funded by Canadian Foundation for Climate and Atmospheric Science and by the Québec provincial consortium URANOS on Climate Change.

REFERENCES

Gilbert J. R., Moler C., and Schreiber R. 1992. Sparse matrices in MATLAB: Design and implementation. *SIAM J. Matrix Anal. Appl.*, 13(1):333—356

Heaps. N. S. 1969. A Two-Dimensional Numerical Sea Model, *Phil. Trans. R. Soc.* A, 265, 93-137.

Pedlosky, J. 1979. Geophysical Fluid Dynamics, Springer Verlag, Berlin and New York.

Shuto N. 1991. Numerical Simulation of Tsunamis --- Its Present and Near Future. *Natural Hazards* 4: 171-191.